AN APPROXIMATE METHOD OF SOLVING HEAT-CONDUCTION PROBLEMS

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The approximate method of [6] is further developed for the example of a temperature field of a plate with a given surface temperature. The error of the solutions obtained is estimated and the possibility of measuring nonstationary heat fluxes is shown.

Statement of the Problem. Approximate mathematical methods are widely used in the analytic theory of heat conduction. Analysis of the solutions shows that they are of asymptotic nature and are appropriate either for short or for long moments of time, while a definite limit of their applicability is not indicated [1-5]. The latter circumstance renders the use of these results rather difficult, and leads to the necessity of searching other approximate methods free of this drawback [6]. Below we develop further the method suggested in [6] for the example of a nonstationary temperature field of a plate with boundary conditions of the first kind, while the surface temperatures can be arbitrary functions of time.

The practical application of the problem under consideration is related to measurement of nonstationary heat fluxes.

The temperature field of the plate is described by the heat-conduction equation

$$\frac{\partial^2 \vartheta}{\partial \bar{x}^2} = \frac{\partial \vartheta}{\partial \mathrm{Fo}} \quad (-1 \leqslant \bar{x} \leqslant 1,\ \mathrm{Fo} \geqslant 0) \tag{1}$$

with the following boundary conditions:

$$\vartheta(-1,\ \mathrm{Fo}) = \vartheta_1(\mathrm{Fo}), \tag{2}$$

$$\vartheta(1,\ \mathrm{Fo}) = \vartheta_2(\mathrm{Fo}), \tag{3}$$

$$\vartheta(\bar{x},\ 0) = 0. \tag{4}$$

For convenience, problem (1)-(4) is written in terms of the excess heat $\vartheta(\bar{x}, \mathrm{Fo}) = t(\bar{x}, \mathrm{Fo}) - t_0$, and the thermophysical properties are assumed to be constant. The solution of (1)-(4) is obtained in several stages.

Initially we study heat-exchange processes for a symmetric plate field with constant surface temperatures. A practically isothermal region with temperature t_0, on which there is no heat-exchange effect at the boundary, is retained at the center of the plate for some initial process duration (Fig. 1). The temperature drop occurs at the portion $L - \delta \leqslant x \leqslant L$, called the nonisothermal region. In terms of some time τ_* the depth of the nonisothermal region δ becomes equal to the plate half width. The temperature field is further varied toward the stationary state. According to [6], the whole process can evolve in time in two stages. The first is characterized by the presence of an isothermal region in the plate and reflects the half-space model in its physical meaning, and the second is characterized by a temperature variation toward the stationary state. The physical meaning of the latter stage is considered below.

First Stage of the Process ($0 < \mathrm{Fo} \leqslant \mathrm{Fo}_*$). At this stage we approximate the plate temperature field in the nonisothermal region by an n-th order parabola [6]. Taking into account the boundary conditions, we obtain

$$\bar{\vartheta}_{\mathrm{I}}(\bar{x},\ \mathrm{Fo}) = \begin{cases} \vartheta_{\mathrm{п}}\left(1 - \dfrac{1-\bar{x}}{\Delta}\right)^n, & 1-\Delta \leqslant \bar{x} \leqslant 1; \\ 0, & 0 \leqslant \bar{x} \leqslant 1-\Delta. \end{cases} \tag{5}$$

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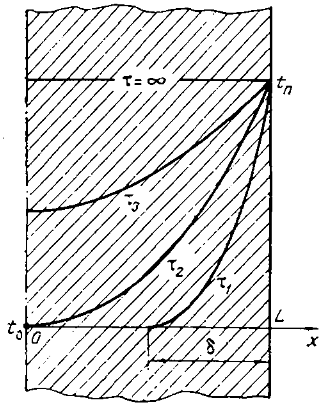


Fig. 1. The time history ($\tau_1 < \tau_2 = \tau_* < \tau_3$) of the plate temperature field.

Introducing the unit function [8]

$$U(z) = \begin{cases} 1, & z > 0 \\ 1/2, & z = 0 \\ 0, & z < 0 \end{cases},$$

expression (5) can be represented more conveniently

$$\bar{\vartheta}_{\rm I}(\bar{x}, {\rm Fo}) = U(\bar{x} - 1 + \Delta)\,\vartheta_{\rm n}\left(1 - \frac{1-\bar{x}}{\Delta}\right)^n. \tag{6}$$

The quantity Δ appearing in (6) depends on Fo, and, as shown in [6], can be obtained by analyzing the heat balance in the nonisothermal region. We briefly consider methods of determining Δ. The accumulated heat is determined by the equation

$$Q = \frac{\lambda}{a} F\delta\bar{\vartheta}_{\rm st},\ \bar{\vartheta}_{\rm st} = \frac{1}{\delta}\int_{L-\delta}^{L} \bar{\vartheta}\,dx.$$

Its change during time $d\tau$

$$dQ = \frac{\lambda}{a}\cdot\frac{F\vartheta_{\rm n}}{n+1}\,d\delta$$

equals the amount of heat withdrawn through the plate surface according to Fourier's law

$$dQ = \lambda\left(\frac{\partial\bar{\vartheta}}{\partial x}\right)_{x=L} F d\tau = \lambda n\,\frac{\vartheta_{\rm n}}{\delta}\,F d\tau.$$

Equating the right-hand sides of the latter equalities and solving the differential equation obtained, we find the function $\delta(\tau)$, written in dimensionless form

$$\Delta = \sqrt{2n(n+1)\,{\rm Fo}}. \tag{7}$$

The first stage is concluded when $\delta = L$, i.e., $\Delta = 1$. The boundary of this stage can then be found from (7)

$$\mathrm{Fo}_* = \frac{a\tau_*}{L^2} = \frac{1}{2n(n+1)}. \tag{8}$$

To complete the solution at the first stage, it remains to determine the parabola index n. We require that (6) best satisfy Eq. (1) in the nonisothermal region. This reduces to the condition [5]

$$\int_{1-\Delta}^{1}\left(\frac{\partial^2\bar{\vartheta}_I}{\partial\bar{x}^2}-\frac{\partial\bar{\vartheta}_I}{\partial \mathrm{Fo}}\right)\bar{\vartheta}_I d\bar{x}=0.$$

Performing the integration, we obtain the quadratic equation

$$2n^2-3n-1=0,$$

whose positive root is n = 1.78. The boundary of this stage $\mathrm{Fo}_* = 0.10$ can be found from (8).

Consider a nonsymmetric plate field at constant surface temperatures $\tilde{\vartheta}(-1, \mathrm{Fo}) = \vartheta_1$, $\tilde{\vartheta}(1, \mathrm{Fo}) = \vartheta_2$. It has been earlier established that the depth of the nonisothermal region is independent of the absolute temperature values, therefore at the moment $\mathrm{Fo}_* = 0.10$ the peaks of the two parabolas coincide at the central point. At the first stage this makes it possible to consider the plate as a set of two half plates, in each of which the temperature field can be described by a function of type (6). Matching the separate expressions, we write the final result

$$\tilde{\vartheta}_I(\bar{x}, \mathrm{Fo}) = U(\bar{x}-1+\Delta)\vartheta_2\left(1-\frac{1-\bar{x}}{\Delta}\right)^n + U(\Delta-1-\bar{x})\vartheta_1\left(1-\frac{1+\bar{x}}{\Delta}\right)^n. \tag{9}$$

At the final moment of the first stage

$$\tilde{\vartheta}_I(\bar{x}, \mathrm{Fo}_*) = U(\bar{x})\vartheta_2\bar{x}^n + U(-\bar{x})\vartheta_1(-\bar{x})^n.$$

Second Stage of the Process ($\mathrm{Fo}_* \leqslant \mathrm{Fo} < \infty$). We seek the temperature field of the plate, starting from the continuity of the solution in passing from the first to the second stage, which allows one to construct the following dependence

$$\tilde{\vartheta}_{II}(\bar{x}, \mathrm{Fo}) = \vartheta_{st}(\bar{x}) - [\vartheta_{st}(\bar{x}) - \tilde{\vartheta}_I(\bar{x}, \mathrm{Fo}_*)]\,\varphi(\mathrm{Fo}-\mathrm{Fo}_*), \tag{10}$$

where $\vartheta_{st}(\bar{x})$ is the stationary solution of the problem under consideration

$$\vartheta_{st}(\bar{x}) = \frac{\vartheta_2-\vartheta_1}{2}\bar{x} + \frac{\vartheta_1+\vartheta_2}{2};$$

and $\varphi(\mathrm{Fo}-\mathrm{Fo}_*)$ is a function on which the conditions $\varphi(0) = 1$, $\varphi(\infty) = 0$ are imposed by physical considerations.

The shape of this function can be found by various arguments. Using, e.g., local potential theory [5], the following expression for φ can be obtained

$$\varphi(\mathrm{Fo}-\mathrm{Fo}_*) = \exp[-\mu(\mathrm{Fo}-\mathrm{Fo}_*)]. \tag{11}$$

To determine the shape of the function φ one can draw upon the theory of the regular regime, as it follows from physical considerations that in the second stage of the process the temperature field is self-similar in time, i.e., it changes similarly to itself.

It is well known [3, 7] that at the regular stage of heat exchange the temperature field is approximately described by the function

$$\vartheta_p(\bar{x}, \mathrm{Fo}) = \vartheta_{st}(\bar{x}) - A_1 f_1(\bar{x})\exp(-\beta_1^2\mathrm{Fo}), \tag{12}$$

where $f_1(\bar{x})$ and A_1 are the first eigenfunction of the problem and a coefficient appearing in it in the exact solution, and β_1 is the first eigenvalue.

We require that at the moment Fo_* expression (12) satisfy the condition

$$\vartheta_p(\bar{x}, \mathrm{Fo}_*) = \tilde{\vartheta}_I(\bar{x}, \mathrm{Fo}_*).$$

Hence

$$A_1 f_1(\bar{x}) = [\vartheta_{st}(\bar{x}) - \tilde{\vartheta}_I(\bar{x}, \mathrm{Fo}_*)]\exp(-\beta_1^2\mathrm{Fo}_*)$$

and (12) is finally transformed to the form

$$\vartheta_p(\bar{x}, \mathrm{Fo}) = \vartheta_{st}(\bar{x}) - [\vartheta_{st}(\bar{x}) - \tilde{\vartheta}_I(\bar{x}, \mathrm{Fo}_*)]\exp[-\beta_1^2(\mathrm{Fo}-\mathrm{Fo}_*)],$$

which fully coincides with (10), (11), with $\mu = \beta_1^2$. It is well known that the first eigenvalue for the plate equals $\pi/2$, while $\mu = (\pi/2)^2$ [1, 3].

Generalizing the results obtained, we write down the calculating equation for the plate temperature field for constant surface temperatures in the time interval $0 < \mathrm{Fo} < \infty$

$$\tilde{\vartheta}(\bar{x},\ \mathrm{Fo}) = U(\mathrm{Fo}_* - \mathrm{Fo})\,\tilde{\vartheta}_{\mathrm{I}}(\bar{x},\ \mathrm{Fo}) + U(\mathrm{Fo} - \mathrm{Fo}_*)\,\tilde{\vartheta}_{\mathrm{II}}(\bar{x},\ \mathrm{Fo}). \tag{13}$$

Varying Boundary Conditions. The solution of the original problem (1)-(4) can be obtained from (13) by means of Duhamel's theorem [1]

$$\vartheta(\bar{x},\ \mathrm{Fo}) = \frac{\partial}{\partial\,\mathrm{Fo}}\int_0^{\mathrm{Fo}}\{U(\mathrm{Fo}_* - \mathrm{Fo} + \mathrm{Fo}')\,\tilde{\vartheta}_{\mathrm{I}}(\bar{x},\ \mathrm{Fo}-\mathrm{Fo}',\ \mathrm{Fo}') + U(\mathrm{Fo} - \mathrm{Fo}' - \mathrm{Fo}_*)\,\tilde{\vartheta}_{\mathrm{II}}(\bar{x},\ \mathrm{Fo}-\mathrm{Fo}',\ \mathrm{Fo}')\}\,d\,\mathrm{Fo}'.$$

It can be shown that for the first stage ($0 < \mathrm{Fo} \leqslant \mathrm{Fo}_*$) the given expression transforms to

$$\vartheta_{\mathrm{I}}(\bar{x},\ \mathrm{Fo}) = \frac{\partial}{\partial\,\mathrm{Fo}}\int_0^{\mathrm{Fo}}\tilde{\vartheta}_{\mathrm{I}}(\bar{x},\ \mathrm{Fo}-\mathrm{Fo}',\ \mathrm{Fo}')\,d\,\mathrm{Fo}', \tag{14}$$

while for the second stage ($\mathrm{Fo}_* \leqslant \mathrm{Fo} < \infty$)

$$\vartheta_{\mathrm{II}}(\bar{x},\ \mathrm{Fo}) = \frac{\partial}{\partial\,\mathrm{Fo}}\left\{\int_{\mathrm{Fo}-\mathrm{Fo}_*}^{\mathrm{Fo}}\tilde{\vartheta}_{\mathrm{I}}(\bar{x},\ \mathrm{Fo}-\mathrm{Fo}',\ \mathrm{Fo}')\,d\,\mathrm{Fo}' + \int_0^{\mathrm{Fo}-\mathrm{Fo}_*}\tilde{\vartheta}_{\mathrm{II}}(\bar{x},\ \mathrm{Fo}-\mathrm{Fo}',\ \mathrm{Fo}')\,d\,\mathrm{Fo}'\right\}. \tag{15}$$

We apply the Leibnitz rule [8] to (14), (15). Finally,

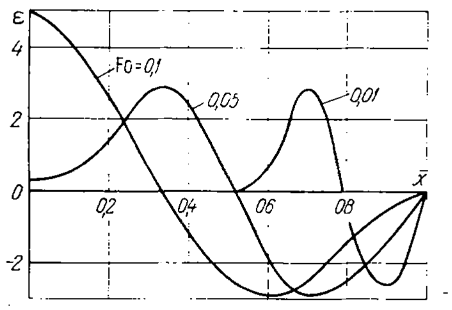
$$\begin{aligned}\vartheta_{\mathrm{I}}(\bar{x},\ \mathrm{Fo}) = \int_0^{\mathrm{Fo}}\Big\{&U[\Delta(\mathrm{Fo}-\mathrm{Fo}') - 1 - \bar{x}]\,\vartheta_1(\mathrm{Fo}')\left[1 - \frac{1+\bar{x}}{\Delta(\mathrm{Fo}-\mathrm{Fo}')}\right]^{n-1}\times\\ &\times(1+\bar{x}) + U[\bar{x} - 1 + \Delta(\mathrm{Fo}-\mathrm{Fo}')]\,\vartheta_2(\mathrm{Fo}')\left[1 - \frac{1-\bar{x}}{\Delta(\mathrm{Fo}-\mathrm{Fo}')}\right]^{n-1}\\ &\times(1-\bar{x})\Big\}\frac{n\sqrt{n(n+1)}\,d\,\mathrm{Fo}'}{\Delta^2(\mathrm{Fo}-\mathrm{Fo}')\sqrt{2(\mathrm{Fo}-\mathrm{Fo}')}};\end{aligned} \tag{16}$$

$$\begin{aligned}\vartheta_{\mathrm{II}}(\bar{x},\ \mathrm{Fo}) = \int_{\mathrm{Fo}-\mathrm{Fo}_*}^{\mathrm{Fo}}\Big\{&U[\Delta(\mathrm{Fo}-\mathrm{Fo}') - 1 - \bar{x}]\,\vartheta_1(\mathrm{Fo}')\Big[1 -\\ &- \frac{1+\bar{x}}{\Delta(\mathrm{Fo}-\mathrm{Fo}')}\Big]^{n-1}(1+\bar{x}) + U[\bar{x} - 1 + \Delta(\mathrm{Fo}-\mathrm{Fo}')]\,\vartheta_2(\mathrm{Fo}')\times\\ &\times\left[1 - \frac{1-\bar{x}}{\Delta(\mathrm{Fo}-\mathrm{Fo}')}\right]^{n-1}(1-\bar{x})\Big\}\frac{n\sqrt{n(n+1)}\,d\,\mathrm{Fo}'}{\Delta^2(\mathrm{Fo}-\mathrm{Fo}')\sqrt{2(\mathrm{Fo}-\mathrm{Fo}')}}\\ &+\mu\int_0^{\mathrm{Fo}-\mathrm{Fo}_*}\Big\{\frac{\vartheta_2(\mathrm{Fo}') - \vartheta_1(\mathrm{Fo}')}{2}\,\bar{x} + \frac{\vartheta_1(\mathrm{Fo}') + \vartheta_2(\mathrm{Fo}')}{2} -\\ &- U(\bar{x})\,\vartheta_2(\mathrm{Fo}')\,\bar{x}^n - U(-\bar{x})\,\vartheta_1(\mathrm{Fo}')(-\bar{x})^n\Big\}\exp[-\mu(\mathrm{Fo}-\mathrm{Fo}'-\mathrm{Fo}_*)]\,d\,\mathrm{Fo}'.\end{aligned} \tag{17}$$

The application of Duhamel's theorem in the sense of integral convergence and differentiation with respect to parameters is justified, since the integrand functions in (14) and (15) are piecewise continuous and increase with time not faster than exponentially (these requirements are imposed on a class of functions to which the Laplace transform, and, consequently, Duhamel's theorem as well, are applicable).

The dependencies obtained can be used to calculate the heat flux in experimental measurements of the plate surface temperature. According to Fourier's law

$$q(\mathrm{Fo}) = -\frac{\lambda}{L}\left.\frac{\partial\vartheta}{\partial\bar{x}}\right|_{\bar{x}=-1},$$



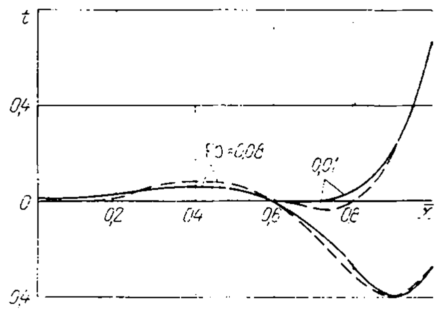


Fig. 2

Fig. 3

Fig. 2. Error analysis in determining the temperature field of a plate, ε, %.

Fig. 3. Comparison of exact and approximate solutions for harmonic variation of the surface temperature.

the thermal flux penetrating the plate through the left surface can be found. At the first stage

$$q_{I}(\mathrm{Fo}) = \frac{\lambda n \sqrt{\mathrm{Fo}_*}}{L}\left\{\frac{\vartheta_1(\mathrm{Fo})}{\sqrt{\mathrm{Fo}}} - \frac{1}{2}\int\limits_0^{\mathrm{Fo}} \frac{\vartheta_1(\mathrm{Fo}') - \vartheta_1(\mathrm{Fo})}{(\mathrm{Fo}-\mathrm{Fo}')^{3/2}}\, d\,\mathrm{Fo}'\right\}, \tag{18}$$

while at the second stage

$$q_{II}(\mathrm{Fo}) = \frac{\lambda n \sqrt{\mathrm{Fo}_*}}{L}\left\{\frac{\vartheta_1(\mathrm{Fo})}{\sqrt{\mathrm{Fo}_*}} - \frac{1}{2}\int\limits_{\mathrm{Fo}-\mathrm{Fo}_*}^{\mathrm{Fo}} \frac{\vartheta_1(\mathrm{Fo}') - \vartheta_1(\mathrm{Fo})}{(\mathrm{Fo}-\mathrm{Fo}')^{3/2}}\, d\,\mathrm{Fo}'\right\} -$$

$$- \frac{\lambda}{L}\mu \int\limits_0^{\mathrm{Fo}-\mathrm{Fo}_*} \left\{\frac{1}{2}\vartheta_2(\mathrm{Fo}') + \left(n - \frac{1}{2}\right)\vartheta_1(\mathrm{Fo}')\right\}$$

$$\exp\left[-\mu(\mathrm{Fo} - \mathrm{Fo}' - \mathrm{Fo}_*)\right] d\,\mathrm{Fo}'. \tag{19}$$

Error Analysis. We estimate the error in calculating the temperature field by the approximate equations (9)-(11) by comparing them with the well-known exact solution [1]. Its magnitude is defined as

$$\varepsilon(\bar{x}, \mathrm{Fo}) = \frac{\vartheta(\bar{x}, \mathrm{Fo}) - \vartheta_{\mathrm{T}}(\bar{x}, \mathrm{Fo})}{\vartheta_{\mathrm{T}\,\max}(\mathrm{Fo})} \cdot 100\%. \tag{20}$$

Results of these calculations are shown in Fig. 2, from which it is seen that the maximum error does not exceed 5%, which is reached in the transition region from one region to the other. At the second stage the error decreases quickly and for Fo = 0.3 it consists of fractions of a percent.

To estimate the possibility of using Eqs. (16), (17), and temperature fields of the plate were calculated for the linear and harmonic variation laws of the surface temperature, while for simplicity the value n = 2 was chosen.

For the linear law $t_p(\mathrm{Fo}) - t_0 = b\mathrm{Fo}$ the solution in the regular stage is

$$\frac{t_{\mathrm{p}}(\mathrm{Fo}) - t(\bar{x}, \mathrm{Fo})}{b} = \frac{1-\bar{x}^2}{\mu} + \frac{1}{3}(1-\bar{x}) + \frac{1}{6}(1-\bar{x})^2\ln(1-\bar{x}) - \frac{1}{4}(1-\bar{x})^2. \tag{21}$$

Its error, defined as the ratio of the difference between the exact and approximate solutions to the maximum temperature drop in the plate, does not exceed 2.5%.

For the harmonic law $t_p(\mathrm{Fo}) = \sin(k\mathrm{Fo})$ (k = 75 was taken in the calculations) the exact (continuous line) and approximate (dashed) solutions are shown in Fig. 3.

Consider the error in determining thermal fluxes. As already noted earlier, the first stage corresponds in its physical meaning to the halfspace model. Equation (18) must coincide with the half-space equation which follows from the exact solution and is of the form [9]

$$q_m(\mathrm{Fo}) = \frac{\lambda}{L\sqrt{\pi}} \left\{ \frac{\vartheta_1(\mathrm{Fo})}{\sqrt{\mathrm{Fo}}} - \frac{1}{2} \int_0^{\mathrm{Fo}} \frac{\vartheta_1(\mathrm{Fo}') - \vartheta_1(\mathrm{Fo})}{(\mathrm{Fo} - \mathrm{Fo}')^{3/2}} \, d\,\mathrm{Fo}' \right\}. \quad (22)$$

Comparing (18) with (22), we see that they differ only in their coefficients, while the deviation is ~0.25%.

To obtain more complete estimates of the error of an approximate solution, it is necessary to compare it with exact solutions for various durations and frequencies of the heat flux.

NOTATION

2L and F, thickness and area of the plate surface; λ and a, thermal conductivity and thermal diffusivity of the material; t_0, initial temperature; t_p, surface temperature of a plate with symmetric heat exchange; t_1 and t_2, temperatures of the left and right surfaces of a plate with nonsymmetric heat exchange; $\bar{\bar{\vartheta}}(\bar{x}, \mathrm{Fo})$, $\tilde{\vartheta}(\bar{x}, \mathrm{Fo})$, excess heatings of a plate under constant boundary conditions for symmetric and nonsymmetric heat exchange; $\bar{x} = x/L$ and $\Delta = \delta/L$, relative coordinate and depth of a nonisothermal region; $\mathrm{Fo} = a\tau/L^2$, Fourier number; τ, time; and $\vartheta_t(\bar{x}, \mathrm{Fo})$, $\vartheta_{t\,max}(\mathrm{Fo})$ exact current and maximum value of excess heating.

LITERATURE CITED

1. A. V. Lykov, Analytical Heat Diffusion Theory, Academic Press, New York (1968).
2. P. V. Tsoi, Methods of Calculating Heat Conduction Problems [in Russian], Énergiya, Moscow (1971).
3. G. M. Kondrat'ev, Regular Thermal Regime [in Russian], Gostekhizdat (1954).
4. L. S. Leibenzon, Selected Papers [in Russian], Vol. 3, Izd. Akad. Nauk SSSR, Moscow (1955).
5. R. S. Shekhter, Variational Methods in Engineering Calculations [Russian translation], Mir, Moscow (1971).
6. A. I. Veinik, Approximate Calculations of Heat Conduction Processes [in Russian], Gosénergoizdat, Moscow–Leningrad (1959).
7. G. N. Dul'nev and É. M. Semyashkin, Heat Exchange in Radioelectronic Instruments [in Russian], Énergiya, Leningrad (1968).
8. G. A. Korn and T. M. Korn, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York (1968).
9. B. Douglas, "Selecting unsteady heat flux sensors," Instrum. Control. Syst., 45, No. 5 (1972).